A generalized ASIP with arrivals to all sites and particle movements in all directions

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Abstract

A generalized n-site Asymmetric Simple Inclusion Process (ASIP) network is studied, where gate-opening instants are determined by a renewal process and arrivals occur to all sites. Various types of batch particle movements between sites are analyzed: (i) unidirectional probabilistic forward movements; (ii) probabilistic forward movements combined with feedback to the first site; and (iii) general probabilistic multidirectional movements. In contrast to the tedious successive substitution method used in previous ASIP studies, an efficient matrix approach is applied to derive the multidimensional probability generating function (PGF) of site occupancies right after gate opening instants. The complexity of the ASIP processes allows us to obtain explicit PGF results for small-size networks only, while for larger networks, a formula to calculate the mean site occupancies is derived for all types of movements. In movement case (i) the means are directly and explicitly calculated. For movement case (ii), where the network is homogeneous with equal probabilities of forward movements from site i to downstream sites $j \geq i$, we show that the ratio between the mean occupancies of consecutive sites approaches a constant when the network becomes large, and calculate this ratio. Finally, we investigate an n-site network where at gate opening instants all gates open simultaneously, and particles move in all directions. Numerical examples are presented.

Keywords: Asymmetric Simple Inclusion Process (ASIP); Generalized ASIP; Particle Movements in all Directions; Matrix Approach for Multi-dimensional PGF; Mean Site Occupancies.

1 Introduction and Outline

The recently introduced Asymmetric Simple Inclusion Process (ASIP) (Reuveni, Eliazar and Yechiali [1, 2, 3]) is a tandem network within the family of Tandem Stochastic Networks (TSNs). It comprises n sites (queues) $Q_1, \ldots, Q_n$ in series, where each site has an unlimited buffer capacity and a gate in front of it. A stochastic process controls gate opening instants. When the gate in front of Q_j opens, all particles present in Q_j instantaneously move to Q_{j+1} $(j = 1, \ldots, n-1)$, and join the cluster of particles already there. When gate n opens, all particles present in Q_n simultaneously leave the system. After a particles transfer, the corresponding gate immediately closes again.

Two notable TSNs are the celebrated Jackson Tandem Networks (TJN) [4, 5, 6, 7] and the Asymmetric Simple Exclusion Process (ASEP) [8, 9, 10]. In the TJN, the buffer capacity of each site is unlimited, while the gate capacity (maximum number of particles that can pass through the gate when it opens) is only a single particle. In the ASEP, both gate capacity and buffer size are limited to a single particle only, causing blocking in particles movement and system evolution. In deviation from the latter two TSNs, in the classical ASIP [1, 2, 3] both buffer size and gate capacity are unlimited, implying that the 'service' mechanism is equivalent to an unlimited batch-size service. In all three TSNs indicated above, particles arrive only to the first site according to a Poisson process and propagate unidirectionally through the system. Gate j opens independently of the other gates every exponential time with parameter μ_j. The classical ASIP model was further studied and analyzed in [11], where occupancy probabilities and limit laws were investigated. Lauber Bonomo and Reuveni [12] calculated occupancy correlations in the classical ASIP.

Boxma, Kella and Yechiali [13] extended the analysis of the ASIP model and investigated a generalized ASIP, where the time intervals between successive gate openings are controlled by a Markovian renewal process, and particles arrive to all sites during the time interval between two successive gate openings. At a gate opening instant only a single gate opens, where gate j opens with probability p_j, $\sum_{j=1}^{n} p_j = 1$. The emphasis in [13] is on the derivation of the probability generating functions (PGFs) of site occupancies. The same authors extended the investigation of [13] and derived the systems' workload distributions [14]. A further analysis of a generalized ASIP (G-ASIP) with arrivals to all sites was performed recently in Yeger and Yechiali [15], where a novel and efficient matrix formulation approach to calculate the joint PGF of site-occupancies was developed, replacing the tedious successive substitution method used in all previous studies. Performance measures of a G-ASIP were studied in [16], where explicit expressions for their distributions and means were obtained for various probability distribution functions of the random intervals between successive gate openings.

As indicated in [1] and in [13], the ASIP model is a generic model that may represent many different stochastic processes in queueing (e.g., simultaneous movements of ships along a series of sluices, crowd movements through sections in an amusement park, car peloton movements along a single-line road, etc.), in physics, in chemistry, in biology, in production lines, and more. In particular, in [17] it was shown that the ASIP model can serve as a mathematical representation of the microbiological process of vesicular transport and transport of proteins between cells.

In this paper, we relax the ASIP and G-ASIP classical assumption of deterministic unidirectional movement of particles between consecutive sites at gate opening instants, and analyze the following three movement variants: (i) Unidirectional probabilistic forward movements; (ii) Probabilistic forward movements combined with feedback to the first site; and (iii) General probabilistic multidirectional movements.

Several key notations are used to describe the dynamics under consideration. We denote site j within the network as Q_j, $(j = 1, 2, \ldots, n)$. The random variable O represents the time elapsing between two consecutive gate opening instants, while at each such instant, gate j opens with probability p_j. A_j counts the random number of arrivals to site j during the time interval between two consecutive gate openings. The multidimensional probability generating function of the number of arrivals during O is denoted by $\hat{A}(z_1, z_2, \ldots, n)$. X_j denotes the number of particles residing at site j right after a gate opening instant, while $G(z_1, z_2, \ldots, z_n) = E[z_1^{X_1} z_2^{X_2} \cdots z_n^{X_n}]$ denotes the multidimensional probability generating function of the site occupancies.

Section 2 details the above different variants of the G-ASIP's particle movements. In Section 3 we apply the matrix approach [15] to calculate the probability generating function (PGF) of site occupancies for movement variant (i) along with mean site occupancies. Movement variant (ii) is studied in Section 4 where the matrix approach is employed to derive the joint PGF of

site occupancies. Due to the complexity of the process [2], explicit PGF results are calculated for small systems only, while for an n-site system, expressions for mean site occupancies are derived. Furthermore, a homogeneous G-ASIP under the latter movement type is studied. It is shown that with equal forward movement probability, m, the occupancy ratio between two successive sites approaches a constant when the number of sites becomes large, namely, $\lim_{n\to\infty} \frac{E[X_{j+1}(n)]}{E[X_j(n)]} = m$, where $X_j(n)$ is the occupancy of site j in an n-site network. In Section 5 we study a G-ASIP with general multidirectional movements and show that the mean site-occupancies vector can be obtained by the formula $E[\vec{X}] = (I-P)^{-1}P \cdot E[\vec{A}]$, where P is the routing probability matrix and $E[\vec{A}]$ is the vector of mean arrivals to the various sites during inter-opening interval O. In Section 6, a G-ASIP is analyzed where at gate opening instants all gates open simultaneously and particles move in all directions. It is shown that the mean site-occupancies vector satisfies $E[\vec{X}] = (I-P)^{-1}P \cdot E[\vec{A}]$ as well, but where, in this case, P is the routing probability matrix corresponding to the general case.

2 Variants of particle movements in the G-ASIP

As indicated in the Introduction, in this paper we extend the research on G-ASIP networks to system's evolution cases not studied before. In the classical ASIP, when the gate in front of site j ($j = 1, 2, \ldots, n-1$) opens all particles present in site j move instantaneously to the subsequent site, site $j+1$, or exit the system when $j = n$. In this paper, the following three variants of particle movements at gate opeining instants are investigated:

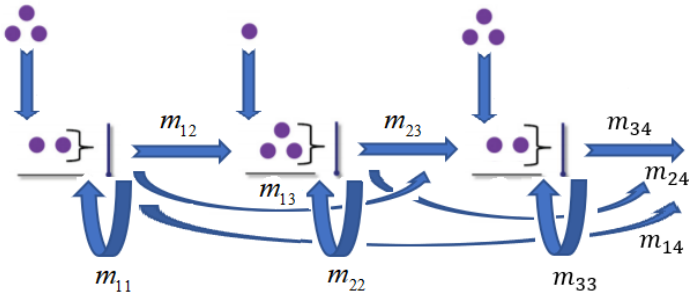
1. Particle probabilistic unidirectional movements to all sites. That is, when gate j opens ($j = 1, 2, \ldots, n$), all particles there move simultaneously with probability m_{jk} to a downstream site k, $k \geq j$, or leave the system with probability $m_{j,n+1}$. This movement scenario is depicted in Figure 1(a) for a 3-site network.

2. Probabilistic forward movements combined with feedback to the first site. When gate j opens ($j = 1, 2, \ldots, n-1$), all particles there move simultaneously either to site $j+1$ (with probability m_j) or fed back to site 1 (with probability $1 - m_j$). When the gate of site n opens, particles either exit the system or fed back to the origin. Figure 1(b) depicts this movement variant for a 3-site network.

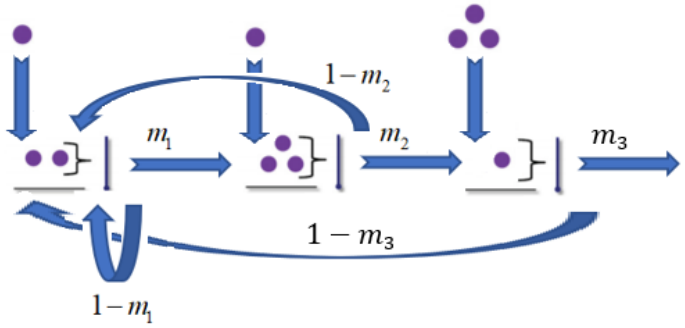
3. General probabilistic multidirectional movements to all sites. That is, when gate j opens ($j = 1, 2, \ldots, n$), all particles there move simultaneously either to site k, $k = 1, 2, \ldots, n$ (with probability m_{jk}), or exit the system (with probability $m_{j,n+1}$). This movement is depicted in Figure 1(c) for a 3-site G-ASIP network.

The first two types of movement dynamics delineate the operational characteristics of a manufacturing assembly, wherein subsequent to each phase, the batch undergoes a quality assessment. The first variant describes a manufacturing assembly where, upon completing a step in the production process, the batch is permitted to progress one or more steps ahead, up to a final completion. The second variant delineates a more stringent manufacturing assembly, wherein at each quality assessment juncture, the batch may have to restart the entire production sequence, due to, for example, a fault. The third ariant extends the movements process to the most generalized case.

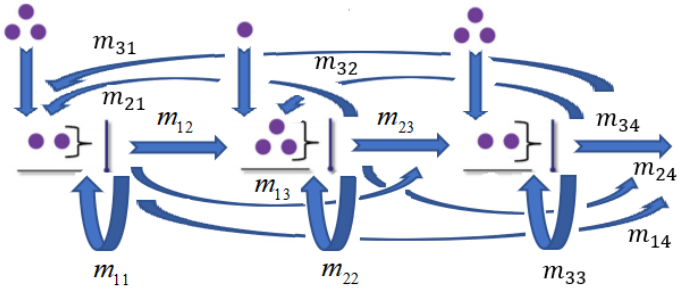
Clearly, variant 3 can be reduced to variant 1 or variant 2 by proper assignment of the probabilities m_{jk}. However, we find it instructive to deal with each one of the variants separately, as under variant 3, only the mean site occupancies are explicitly derived.



(a) Particle probabilistic unidirectional movements to all sites, where all particles in site j move simultaneously with probability m_{jk} to site k $(k \geq j)$ or leave the system with probability m_{j4}.



(b) Paticle movement with feedback to the first site, where all particles in site j move simultaneously either to site $j+1$ (with probability m_j) or fed back to site 1 (with probability $1-m_j$).



(c) General probabilistic multidirectional movements to all sites, where all particles in site j move simultaneously either to site k $(k = 1, 2, , 3)$ (with probability m_{jk}), or exit the system (with probability m_{j4}).

Figure 1: Variants of particle movements within a 3-site G-ASIP

3 Unidirectional G-ASIP with probabilistic movements to all sites

Consider an n-site G-ASIP network, where gate openings are governed by a renewal process. At a gate opening instant only a single gate opens and it is gate j with probability $p_j > 0$, $\sum_{j=1}^{n} p_j = 1$.

The gate inter-opening intervals are i.i.d random variables, all distributed like O. During a gate inter-opening interval O, arrivals occur at all sites $Q_1, Q_2, \ldots, Q_n$ according to a general arrival process. Let $\hat{A}(z_1, z_2, \ldots, z_n)$ denote the joint PGF of the number of arrivals to the various sites during an inter-opening interval. The complexity of the model (see [2, 13]) allows us to derive explicit solutions for the site occupancies joint PGF of small size networks only. For the general n-site system, we derive a matrix formulation to calculate the mean site occupancies and present a direct recursive formula for that calculation. As a numerical example, we consider a homogeneous G-ASIP with $n = 10$ sites and calculate explicitly the system's mean site occupancies.

3.1 Probabilistic forward movements with $n = 2$ sites

We consider a 2-site G-ASIP network with probabilistic forward movements as follows: At gate opening instant, gate j opens with probability p_j, $j = 1, 2$, where $p_1 + p_2 = 1$. When the gate at the first site opens, all particles there either remain in Q_1 with probability m_{11}, or move forward to Q_2 with probability m_{12}, or leave the system with probability m_{13}, where $m_{11} + m_{12} + m_{13} = 1$. When the gate of site Q_2 opens, all particles there either remain in Q_2 with probability m_{22}, or leave the system with probability m_{23}, where $m_{22} + m_{23} = 1$. The particles dynamic movement is depicted in Figure 2:

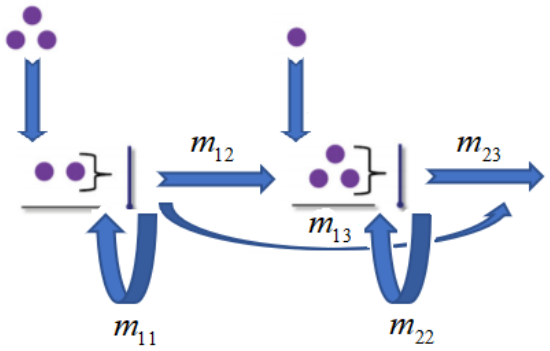


Figure 2: Particle dynamic probabilistic movements in a 2-site network: particles from site 1 can move to site 2 with probability m_{12}, or exit the system with probability m_{13}, or stay in site 1 with probability $m_{11} = 1 - (m_{12} + m_{13})$. Particles from site 2 can either exit the system with probability m_{23} or stay in site 2 with probability $m_{22} = 1 - m_{23}$. Arrivals occur at each one of the sites.

Let X_j, $j = 1, 2$, denote the number of particles (occupancy) at site j right after a gate opening, and let X_j^r denote the number of particles in Q_j right after the rth gate opening. Let $A_j^{(r)}$ be the number of arrivals to site j during the interval between the $(r-1)^{th}$ and the rth gate openings. Define $G(z_1, z_2) = E[z_1^{X_1} z_2^{X_2}]$, $|z_j| \leq 1$ as the joint PGF of the site occupancies right after a gate opening instant.

3.1.1 Derivation of the PGF $G(z_1, z_2)$

The two-dimensional laws of motion are:

If gate 1 opens at the $(r+1)^{\text{th}}$ gate opening, then

$$\begin{aligned}
\text{with probability } m_{11}: \quad & X_1^{(r+1)} = X_1^{(r)} + A_1^{(r+1)}, \\
& X_2^{(r+1)} = X_2^{(r)} + A_2^{(r+1)}, \\
\text{with probability } m_{12}: \quad & X_1^{(r+1)} = 0, \\
& X_2^{(r+1)} = X_1^{(r)} + A_1^{(r+1)} + X_2^{(r)} + A_2^{(r+1)}, \\
\text{with probability } m_{13}: \quad & X_1^{(r+1)} = 0, \\
& X_2^{(r+1)} = X_2^{(r)} + A_2^{(r+1)}.
\end{aligned}$$

If gate 2 opens at step $r+1$, then

$$\begin{aligned}
\text{with probability } m_{22}: \quad & X_1^{(r+1)} = X_1^{(r)} + A_1^{(r+1)}, \\
& X_2^{(r+1)} = X_2^{(r)} + A_2^{(r+1)}, \\
\text{with probability } m_{23}: \quad & X_1^{(r+1)} = X_1^{(r)} + A_1^{(r+1)}, \\
& X_2^{(r+1)} = 0.
\end{aligned}$$

In steady state, when $A_j^{(r)} \to A_j$ we obtain:

$$\begin{aligned}
G(z_1,z_2) =& E[z_1^{X_1} z_2^{X_2}] \\
=& p_1 m_{11} E[z_1^{X_1+A_1} z_2^{X_2+A_2}] + p_1 m_{12} E[z_1^0 z_2^{X_1+A_1+X_2+A_2}] + p_1 m_{13} E[z_1^0 z_2^{X_2+A_2}] \\
& + p_2 m_{22} E[z_1^{X_1+A_1} z_2^{X_2+A_2}] + p_2 m_{23} E[z_1^{X_1+A_1} z_2^0] \qquad (1) \\
=& p_1 m_{11} \hat{A}(z_1,z_2) G(z_1,z_2) + p_1 m_{12} \hat{A}(z_2,z_2) G(z_2,z_2) + p_1 m_{13} \hat{A}(1,z_2) G(1,z_2) \\
& + p_2 m_{22} \hat{A}(z_1,z_2) G(z_1,z_2) + p_2 m_{23} \hat{A}(z_1,1) G(z_1,1).
\end{aligned}$$

Hence,

$$\begin{aligned}
G(z_1,z_2) =& p_1 m_{11} \hat{A}(z_1,z_2) G(z_1,z_2) + p_1 m_{12} \hat{A}(z_2,z_2) G(z_2,z_2) + p_1 m_{13} \hat{A}(1,z_2) G(1,z_2) \\
& + p_2 m_{22} \hat{A}(z_1,z_2) G(z_1,z_2) + p_2 m_{23} \hat{A}(z_1,1) G(z_1,1).
\end{aligned} \qquad (2)$$

The derivation of the PGF $G(z_1,z_2)$ from Eq. (2) by proper substitutions is detailed in Appendix

A, which yields Eq. (3) below

$$
\begin{aligned}
&G(z_1,z_2) = \\
&\frac{p_1m_{12}\hat{A}(z_2,z_2)}{1-(p_1m_{11}+p_2m_{22})\hat{A}(z_1,z_1)}\times \\
&\frac{p_1m_{13}p_2m_{23}\hat{A}(1,z_2)+\big(1-(p_1m_{11}+p_1m_{13}+p_2m_{22})\hat{A}(1,z_2)\big)p_2m_{23}\hat{A}(z_2,1)\frac{p_1m_{12}+p_1m_{13}}{1-(p_1m_{11}+p_2)\hat{A}(z_2,1)}}{\big(1-(p_1(1-m_{13})+p_2m_{22})\hat{A}(z_2,z_2)\big)\big(1-(p_1(1-m_{12})+p_2m_{22})\hat{A}(1,z_2)\big)-p_1m_{13}\hat{A}(1,z_2)p_1m_{12}\hat{A}(z_2,z_2)} \\
&+\frac{p_1m_{13}\hat{A}(1,z_2)}{1-(p_1m_{11}+p_2m_{22})\hat{A}(z_1,z_2)}\times \\
&\frac{\big(1-(p_1m_{11}+p_1m_{12}+p_2m_{22})\hat{A}(z_2,z_2)\big)p_2m_{23}+p_1m_{12}\hat{A}(z_2,z_2)p_2m_{23}\hat{A}(z_2,1)\frac{p_1m_{12}+p_1m_{13}}{1-(p_1m_{11}+p_2)\hat{A}(z_2,1)}}{\big(1-(p_1(1-m_{13})+p_2m_{22})\hat{A}(z_2,z_2)\big)\big(1-(p_1(1-m_{12})+p_2m_{22})\hat{A}(1,z_2)\big)-p_1m_{13}\hat{A}(1,z_2)p_1m_{12}\hat{A}(z_2,z_2)} \\
&+\frac{p_2m_{23}\hat{A}(z_1,1)}{1-(p_1m_{11}+p_2m_{22})\hat{A}(z_1,z_2)}\frac{p_1m_{12}+p_1m_{13}}{1-(p_1m_{11}+p_2)\hat{A}(z_1,1)}.
\end{aligned}
\tag{3}
$$

It is clear that for larger values of n, the calculation of the PGF becomes extremely complex.

Remark 1. *A Special Case*

Substituting $m_{11}=0$, $m_{12}=1$, $m_{13}=0$, $m_{22}=0$ and $m_{23}=1$ in Eq. (3) and letting arrivals occur at the first site only such that $\hat{A}(z_i,z_j)=\hat{A}(z_i,1)=\hat{A}(z_i)$, then the result coincides with Eq. (32) in [13], namely,

$$
G(z_1,z_2)=\frac{p_1p_2\hat{A}(z_2)}{1-p_2\hat{A}(z_2)}\frac{p_1\hat{A}(z_2)}{1-p_1\hat{A}(z_2)}+\frac{p_1p_2\hat{A}(z_1)}{1-p_2\hat{A}(z_1)}. \tag{4}
$$

Note: One can tediously calculate PGFs of n-site networks following the matrix formulation approach detailed in Appendix A. Evidently, the resulting expressions will turn highly complex. Yet, if the interest is in calculating mean site occupancies, the computational effort is considerably lower as will be shown in Section 3.2 below.

3.2 Mean site occupancies for an n-site network

By concentrating on mean site occupancies, a direct approach can be adopted. The occupancies of sites in networks with $n=2$ and $n=3$ are calculated explicitly in Appendix B. Subsequently, we proceed to calculate the mean site occupancies of an n-site network directly from the laws of motion:

$$
\begin{pmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{pmatrix} = \begin{pmatrix} 1-p_1(1-m_{11}) & 0 & \cdots & 0 \\ p_1m_{12} & 1-p_2(1-m_{22}) & & 0 \\ \vdots & & \ddots & \\ p_1m_{1n} & p_2m_{2n} & \cdots & 1-p_n(1-m_{nn}) \end{pmatrix} \begin{pmatrix} E[X_1]+E[A_1] \\ E[X_2]+E[A_2] \\ \vdots \\ E[X_n]+E[A_n] \end{pmatrix}, \tag{5}
$$

which leads to the following matrix representation solution

$$\begin{pmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{pmatrix} = \begin{pmatrix} p_1(1-m_{11}) & 0 & \cdots & 0 \\ -p_1 m_{12} & p_2(1-m_{22}) & & 0 \\ \vdots & & \ddots & \\ -p_1 m_{1n} & -p_2 m_{2n} & \cdots & p_n(1-m_{nn}) \end{pmatrix}^{-1} \times \begin{pmatrix} 1-p_1(1-m_{11}) & 0 & \cdots & 0 \\ p_1 m_{12} & 1-p_2(1-m_{22}) & & 0 \\ \vdots & & \ddots & \\ p_1 m_{1n} & p_2 m_{2n} & \cdots & 1-p_n(1-m_{nn}) \end{pmatrix} \begin{pmatrix} E[A_1] \\ E[A_2] \\ \vdots \\ E[A_n] \end{pmatrix}. \quad (6)$$

Further algebra results in an explicit recursive formula for the mean occupancies of the sites, namely,

$$E[X_k] = \frac{1}{p_k(1-m_{kk})} \sum_{j=1}^{k-1} p_j m_{jk}(E[X_j]+E[A_j]) + \frac{1}{p_k(1-m_{kk})} E[A_k] - E[A_k], \quad k = 1, 2, \cdots, n. \quad (7)$$

3.2.1 Mean site occupancies for an n-site network in a homogeneous G-ASIP

A homogeneous G-ASIP where all gates open with equal probabilities $p_j = \frac{1}{n}$, $j = 1, \cdots, n$ has been shown in [16] to be optimal with respect to the following measures of efficiency: (i) minimizing the mean overall load subject to a given aggregate rate; (ii) minimizing the variance of the overall load subject to a given aggregate rate; (iii) maximizing the probability of a zero overall load, i.e., the probability of an empty system, subject to a given aggregate rate; and (iv) minimizing the variance of the overall load subject to a given mean overall load. As an illustration, we calculate the mean site occupancies of a 10-site homogeneous ASIP where $p_j = \frac{1}{10}$ and where forward movement probabilities are also evenly distributed, that is, $m_{ij} = \frac{1}{12-i}$, $i = 1, \cdots, 10$, $i \le j \le 11$. Eq. (6) becomes

$$\begin{pmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_{10}] \end{pmatrix} = \begin{pmatrix} \frac{1}{11} & 0 & \cdots & 0 \\ -\frac{1}{100} & \frac{9}{100} & & 0 \\ \vdots & & \ddots & \\ -\frac{1}{100} & -\frac{1}{100} & \cdots & \frac{1}{20} \end{pmatrix}^{-1} \begin{pmatrix} \frac{10}{11} & 0 & \cdots & 0 \\ \frac{1}{110} & \frac{91}{100} & & 0 \\ \vdots & & \ddots & \\ \frac{1}{110} & \frac{1}{100} & \cdots & \frac{19}{20} \end{pmatrix} \begin{pmatrix} E[A_1] \\ E[A_2] \\ \vdots \\ E[A_{10}] \end{pmatrix}. \quad (8)$$

That is

$$\begin{pmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_{10}] \end{pmatrix} = \begin{pmatrix} 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.11 & 10.11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.25 & 1.25 & 10.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.43 & 1.43 & 1.43 & 10.43 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.67 & 1.67 & 1.67 & 1.67 & 10.67 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 11 & 0 & 0 & 0 & 0 \\ 2.5 & 2.5 & 2.5 & 2.5 & 2.5 & 2.5 & 11.5 & 0 & 0 & 0 \\ 3.33 & 3.33 & 3.33 & 3.33 & 3.33 & 3.33 & 3.33 & 12.33 & 0 & 0 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 14 & 0 \\ 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 19 \end{pmatrix} \begin{pmatrix} E[A_1] \\ E[A_2] \\ \vdots \\ E[A_{10}] \end{pmatrix}. \quad (9)$$

In the case that during the time interval O the mean total number of arrivals to the system is 1, and the arrivals are distributed uniformly between the sites, such that $E[A_j] = \bar{A} = \frac{1}{n} = \frac{1}{10}$, the

occupancies of a 10-site system are analytically calculated by the formula above and compared to a Monte-Carlo simulation. The comparison is depicted in Figure 3:

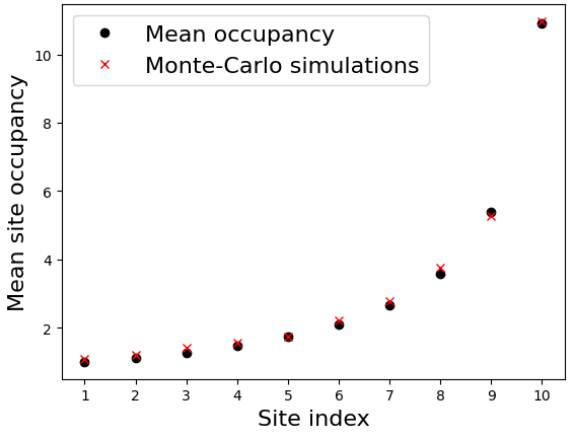


Figure 3: Mean site occupancies calculated and compared to a Monte-Carlo simulation for probabilistic forward movements where $E[A_j] = \frac{1}{10}, m_{ij} = \frac{1}{12-i}$. The circle markers represent the calculated values according to Eq. (9), while the cross markers depict the Monte-Carlo simulations.

4 Probabilistic forward movements with feedback to the first site

Consider an n-site tandem Jackson-type network with feedback to the first node [18], where single arrivals occur at the first site only. A tandem network with feedback to the first site may represent a production process where a product undergoes n tasks in sequence before completion. When a task fails, the entire sequence of tasks must be repeated. A G-ASIP with feedback to the first site similarly represents an n-site batch-production network where, after an opening of gate j, the batch undergoes inspection. The inspection is found 'good' with probability m_j and the batch moves forward unidirectional to be processed in the next site. Otherwise, with probability $1 - m_j$, the batch fails the inspection and is fed back to the first site to start the entire process all over again. We analyze a corresponding G-ASIP network with general arrival process at all sites during intervals between gates openings and with feedback to the first site, and derive explicitly the PGF of site occupancies for a network with $n = 2$ sites. We also study the mean site occupancies of an n-site system. In an $n = 2$ site system, if gate 1 opens particles move forward from Q_1 to Q_2 with probability m_1, or fed back with probability $1 - m_1$. If gate 2 opens particles either exit the system with probability m_2, or fed back to Q_1 with the complementary probability $1 - m_2$. The particles dynamic movement is depicted in Figure 4:

4.1 Derivation of $G(z_1, z_2)$ using matrix representation

The two-dimensional laws of motion are:

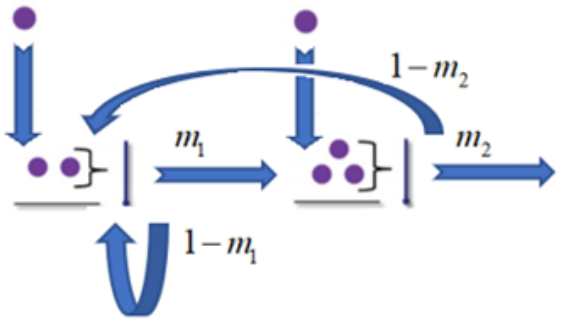


Figure 4: Particle movements with feedback to the first site in a 2-site network: particles from site 1 can move to site 2 with probability m_1, or remain in the site with probability $1-m_1$. Particles from site 2 can either exit the system with probability m_2 or feedback to the first site with the complementary probability $1-m_2$. Arrivals occur at each one of the sites.

If gate 1 opens at step $r+1$, then

$$\begin{aligned}
\text{with probability } m_{11}: \quad & X_1^{(r+1)} = 0, \\
& X_2^{(r+1)} = X_1^{(r)} + A_1^{(r+1)} + X_2^{(r)} + A_2^{(r+1)}, \\
\text{with probability } 1-m_{11}: \quad & X_1^{(r+1)} = X_1^{(r)} + A_1^{(r+1)}, \\
& X_2^{(r+1)} = X_2^{(r)} + A_2^{(r+1)}.
\end{aligned}$$

If gate 2 opens at step $r+1$, then

$$\begin{aligned}
\text{with probability } m_2: \quad & X_1^{(r+1)} = X_1^{(r)} + A_1^{(r+1)}, \\
& X_2^{(r+1)} = 0, \\
\text{with probability } 1-m_2: \quad & X_1^{(r+1)} = X_1^{(r)} + A_1^{(r+1)} + X_2^{(r)} + A_2^{(r+1)}, \\
& X_2^{(r+1)} = 0.
\end{aligned}$$

In steady state we obtain

$$\begin{aligned}
G(z_1,z_2) =& E[z_1^{X_1} z_2^{X_2}] \\
=& p_1 m_1 E[z_1^0 z_2^{X_1+A_1+X_2+A_2}] + p_1(1-m_1) E[z_1^{X_1+A_1} z_2^{X_2+A_2}] + p_2 m_2 E[z_1^{X_1+A_1} z_2^0] \\
&+ p_2(1-m_2) E[z_1^{X_1+A_1+X_2+A_2} z_2^0] \\
=& p_1 m_1 \hat{A}(z_2,z_2) G(z_2,z_2) + p_1(1-m_1)\hat{A}(z_1,z_2) G(z_1,z_2) \\
&+ p_2 m_2 \hat{A}(z_1,1) G(z_1,1) + p_2(1-m_{22})\hat{A}(z_1,z_1) G(z_1,z_1).
\end{aligned} \tag{10}$$

Hence,

$$\begin{aligned}
G(z_1,z_2) =& p_1 m_1 \hat{A}(z_2,z_2) G(z_2,z_2) + p_1(1-m_1)\hat{A}(z_1,z_2) G(z_1,z_2) + p_2 m_2 \hat{A}(z_1,1) G(z_1,1) \\
&+ p_2(1-m_2)\hat{A}(z_1,z_1) G(z_1,z_1),
\end{aligned} \tag{11}$$

leading to

$$G(z_1,z_2)=\frac{p_1m_1\hat{A}(z_2,z_2)G(z_2,z_2)+p_2m_2\hat{A}(z_1,1)G(z_1,1)+p_2(1-m_2)\hat{A}(z_1,z_1)G(z_1,z_1)}{1-p_1(1-m_1)\hat{A}(z_1,z_2)}. \quad (12)$$

The full derivation of the PGF $G(z_1,z_2)$ from Eq. (12) is detailed in Appendix C. We obtain

$$\begin{aligned}
&G(z_1,z_2)=\\
&\frac{p_1m_1\hat{A}(z_2,z_2)}{1-p_1(1-m_1)\hat{A}(z_1,z_2)}\times\\
&\frac{p_1m_1p_2m_2\hat{A}(z_2,1)}{\big(1-(p_1+p_2(1-m_2)\hat{A}(z_2,z_2)\big)\big(1-(p_1(1-m_1)+p_2m_2)\hat{A}(z_2,1)\big)-p_2^2m_2(1-m_2)\hat{A}(z_2,z_2)\hat{A}(z_2,1)}\\
&+\frac{p_2m_2\hat{A}(z_1,1)}{1-p_1(1-m_1)\hat{A}(z_1,z_2)}\times\\
&\frac{p_1m_1(1-(p_1+p_2(1-m_2))\hat{A}(z_1,z_1)}{\big(1-(p_1+p_2(1-m_2)\hat{A}(z_1,z_1)\big)\big(1-(p_1(1-m_1)+p_2m_2)\hat{A}(z_1,1)\big)-p_2^2m_2(1-m_2)\hat{A}(z_1,z_1)\hat{A}(z_1,1)}\\
&+\frac{p_2(1-m_2)\hat{A}(z_1,z_1)}{1-p_1(1-m_1)\hat{A}(z_1,z_2)}\times\\
&\frac{p_1m_1p_2m_2\hat{A}(z_1,1)}{\big(1-(p_1+p_2(1-m_2)\hat{A}(z_1,z_1)\big)\big(1-(p_1(1-m_1)+p_2m_2)\hat{A}(z_1,1)\big)-p_2^2m_2(1-m_2)\hat{A}(z_1,z_1)\hat{A}(z_1,1)}.
\end{aligned} \quad (13)$$

4.2 Mean site occupancies for an n-site network

From the laws of motion

$$\begin{pmatrix}E[X_1]\\E[X_2]\\\vdots\\E[X_n]\end{pmatrix}=\begin{pmatrix}1-p_1m_1 & p_2(1-m_2) & p_3(1-m_3) & \cdots & p_n(1-m_n)\\ p_1m_1 & 1-p_2 & 0 & \cdots & 0\\ 0 & p_2m_2 & 1-p_3 & & \\ & \vdots & & \ddots & \\ 0 & \cdots & 0 & p_{n-1}m_{n-1} & 1-p_n\end{pmatrix}\begin{pmatrix}E[X_1]+E[A_1]\\E[X_2]+E[A_2]\\\vdots\\E[X_n]+E[A_n]\end{pmatrix}, \quad (14)$$

The above leads to a matrix representation solution

$$\begin{aligned}\begin{pmatrix}E[X_1]\\E[X_2]\\\vdots\\E[X_n]\end{pmatrix}=&\begin{pmatrix}p_1m_1 & -p_2(1-m_2) & -p_3(1-m_3) & \cdots & -p_n(1-m_n)\\ -p_1m_1 & p_2 & 0 & \cdots & 0\\ 0 & -p_2m_2 & p_3 & & \\ & \vdots & & \ddots & \\ 0 & \cdots & 0 & -p_{n-1}m_{n-1} & p_n\end{pmatrix}^{-1}\\ &\times\begin{pmatrix}1-p_1m_1 & p_2(1-m_2) & p_3(1-m_3) & \cdots & p_n(1-m_n)\\ p_1m_1 & 1-p_2 & 0 & \cdots & 0\\ 0 & p_2m_2 & 1-p_3 & & \\ & \vdots & & \ddots & \\ 0 & \cdots & 0 & p_{n-1}m_{n-1} & 1-p_n\end{pmatrix}\begin{pmatrix}E[A_1]\\E[A_2]\\\vdots\\E[A_n]\end{pmatrix}.\end{aligned} \quad (15)$$

4.3 A special case in a homogeneous G-ASIP

As a consequence of the universal properties of homogenous G-ASIP indicated in section 3.2, we consider an n-site homogeneous G-ASIP with feedback to the first site, where the probability that gate j opens is $p_j = \frac{1}{n}$, $j = 1, \ldots, n$, and the uniform non-failure (success) probabilities are $m_j = m$ for all sites.

Proposition 4.1. *Mean site occupancies in a homogeneous G-ASIP with uniform success probabilities are given by*

$$\begin{pmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{pmatrix} = \begin{pmatrix} \frac{n-m^n}{m^n} & \frac{n(1-m^{n-1})}{m^n} & \frac{n(1-m^{n-2})}{m^n} & \cdots & \frac{n(1-m)}{m^n} \\ \frac{n}{m^{n-1}} & \frac{n-m^{n-1}}{m^{n-1}} & \frac{n(1-m^{n-2})}{m^{n-1}} & \cdots & \frac{n(1-m)}{m^{n-1}} \\ \frac{n}{m^{n-2}} & \cdots & \frac{n-m^{n-2}}{m^{n-2}} & \cdots & \frac{n(1-m)}{m^{n-2}} \\ & \vdots & & \ddots & \\ \frac{n}{m} & \cdots & & \frac{n}{m} & \frac{n-m}{m} \end{pmatrix} \begin{pmatrix} E[A_1] \\ E[A_2] \\ \vdots \\ E[A_n] \end{pmatrix}. \tag{16}$$

The proof for Proposition 4.1 is detailed in Appendix D.

Corollary 4.1.1. *The ratio between mean site occupancies of consecutive sites in a homogeneous G-ASIP with uniform success probabilities $m_j = m$ satisfies, for all j,*

$$\lim_{n\to\infty} \frac{E[X_{j+1}(n)]}{E[X_j(n)]} = m.$$

In other words (suppressing n in $E[X_j(n)]$), when $n \to \infty$, $E[X_j] = m^{j-1}E[X_1]$.

The proof for Corollary 4.1.1 is detailed in Appendix E.

4.4 Numerical examples

The mean site occupancies are numerically calculated for a homogeneous 10-site G-ASIP where arrivals are evenly distributed among the sites, $E[A_j] = \bar{A}$, for three different inspection probabilities: (i) Equal success and failure probabilities $m_j = \frac{1}{2}$, $j = 1, \ldots, 10$; (ii) Uniform low failure probabilities $m_j = \frac{7}{10}$, $j = 1, \ldots, 10$; and (iii) Decreasing failure probabilities $1 - m_j = \frac{1}{j+1}$, $j = 1, \ldots, 10$.

4.4.1 Equal success and failure probabilities

Substituting in Eq. (16) the values $p_j = \frac{1}{10}$, $j = 1, \ldots, 10$ and $m_j = \frac{1}{2}$ gives

$$\begin{pmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_{10}] \end{pmatrix} = \begin{pmatrix} 10,239 & 10,220 & 10,200 & 10,160 & 10,080 & 9920 & 9600 & 8960 & 7680 & 5120 \\ 5120 & 5119 & 5100 & 5080 & 5040 & 4960 & 4800 & 4480 & 3840 & 2560 \\ 2560 & 2560 & 2559 & 2540 & 2520 & 2480 & 2400 & 2240 & 1920 & 1280 \\ 1280 & 1280 & 1280 & 1279 & 1260 & 1240 & 120 & 1120 & 960 & 640 \\ 640 & 640 & 640 & 640 & 639 & 620 & 600 & 560 & 480 & 320 \\ 320 & 320 & 320 & 320 & 320 & 319 & 300 & 280 & 240 & 160 \\ 160 & 160 & 160 & 160 & 160 & 160 & 159 & 140 & 120 & 80 \\ 80 & 80 & 80 & 80 & 80 & 80 & 80 & 79 & 60 & 40 \\ 40 & 40 & 40 & 40 & 40 & 40 & 40 & 40 & 39 & 20 \\ 20 & 20 & 20 & 20 & 20 & 20 & 20 & 20 & 20 & 19 \end{pmatrix} \begin{pmatrix} E[A_1] \\ E[A_2] \\ \vdots \\ E[A_{10}] \end{pmatrix}. \tag{17}$$

Similar to the case in Section 3.2 where $E[A_j] = \bar{A} = \frac{1}{n} = \frac{1}{10}$, the occupancies of a 10-site system are analytically calculated and compared to a Monte-Carlo simulation. The comparison is depicted in Figure 5 on a log scale, where it is seen that $\lim_{n\to\infty} \frac{E[X_{i+1}(n)]}{E[X_i(n)]} = \frac{1}{2} = m_j$.

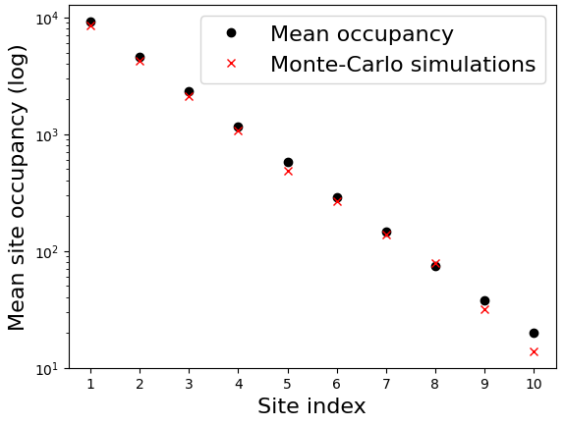


Figure 5: Mean site occupancies calculated and compared to a Monte-Carlo simulation for Feedback to the first site where $m_j = \frac{1}{2}$, $j = 1, \ldots, 10$ in log scale which shows the property $\lim_{n\to\infty} \frac{E[X_{j+1}(n)]}{E[X_j(n)]} = \frac{1}{2}$. The circle markers represent the calculated values according to Eq. (17), while the cross markers depict the Monte-Carlo simulations.

4.4.2 Uniform feedback probabilities

Substituting $p_j = \frac{1}{10}$, $j = 1, \ldots, 10$ and $m_j = 0.7$ in Eq. (16) gives

$$\begin{pmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_{10}] \end{pmatrix} = \begin{pmatrix} 27.68 & 17.57 & 16.33 & 14.96 & 13.44 & 11.74 & 9.86 & 7.77 & 5.45 & 2.87 \\ 25.81 & 24.81 & 14.7 & 13.47 & 12.09 & 10.57 & 8.88 & 6.99 & 4.9 & 2.58 \\ 23.23 & 23.23 & 22.23 & 12.12 & 10.88 & 9.51 & 7.99 & 6.3 & 4.41 & 2.32 \\ 20.9 & 20.9 & 20.9 & 19.9 & 9.8 & 8.56 & 7.19 & 5.67 & 3.97 & 2.09 \\ 18.82 & 18.82 & 18.82 & 18.82 & 17.82 & 7.7 & 6.47 & 5.1 & 3.57 & 1.88 \\ 16.94 & 16.94 & 16.94 & 16.94 & 16.94 & 15.94 & 5.82 & 4.58 & 3.28 & 1.69 \\ 15.24 & 15.24 & 15.24 & 15.24 & 15.24 & 15.24 & 14.24 & 4.13 & 2.9 & 1.52 \\ 13.72 & 13.72 & 13.72 & 13.72 & 13.72 & 13.72 & 13.72 & 12.72 & 2.6 & 1.37 \\ 12.35 & 12.35 & 12.35 & 12.35 & 12.35 & 12.35 & 12.35 & 12.35 & 11.35 & 1.23 \\ 11.11 & 11.11 & 11.11 & 11.11 & 11.11 & 11.11 & 11.11 & 11.11 & 11.11 & 10.11 \end{pmatrix} \begin{pmatrix} E[A_1] \\ E[A_2] \\ \vdots \\ E[A_{10}] \end{pmatrix}. \tag{18}$$

When arrivals are evenly distributed between the sites, such that on average there is one arrival during the time interval between two successive gate openings, then $E[A_j] = \bar{A} = \frac{1}{n} = \frac{1}{10}$. The occupancies of a 10-site system are analytically calculated and compared to a Monte-Carlo simulation. The comparison is depicted on a log sacle in Figure 6 where it exhibits that $\lim_{n\to\infty} \frac{E[X_{j+1}(n)]}{E[X_j(n)]} = 0.7$.

4.4.3 Decreasing failure probabilities

In this subsection we depict in Figure 7 the mean site occupancies in the case of decreasing failure probabilities for a 10-site network and compare it to a Monte-Carlo simulation. Substituting in

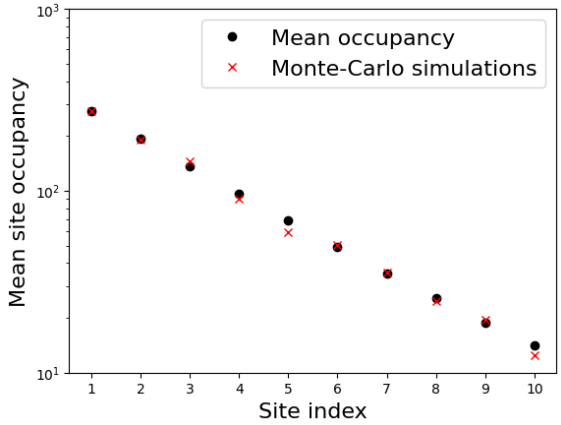


Figure 6: Mean site occupancies calculated and compared to a Monte-Carlo simulations for Feedback to the first site where $m_j = 0.7$, $j = 1, \ldots, 10$. It is seen that $\lim_{n\to\infty} \frac{E[X_{j+1}(n)]}{E[X_j(n)]} = 0.7$. The circle markers represent the calculated values according to Eq. (18), while the cross markers depict the Monte-Carlo simulations.

Eq. (15) $p_j = \frac{1}{10}$, $j = 1, \ldots, 10$, and decreasing failure probabilities $1 - m_j = \frac{1}{j+1}$, while taking $E[A_j] = \bar{A} = \frac{1}{n} = \frac{1}{10}$, we obtain Figure 7.

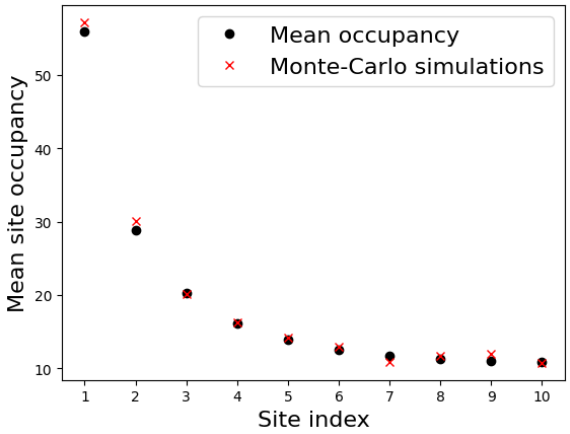


Figure 7: Mean site occupancies for feedback to the first site in a 10-site system with decreasing failure probabilities $1 - m_j = \frac{1}{j+1}$, $j = 1, \ldots, 10$.

5 General ASIP with movements in all directions

Consider a G-ASIP network, where arrivals occur at all sites and a renewal process controls the instants of gate openings. The times between two consecutive gate openings are i.i.d random variables all distributed like O as defined above. At a gate opening of Q_i all particles present there either

move instantaneously to Q_j $(j = 1, 2, \ldots, n)$ with probability p_{ij}, or exit the system with probability $p_{i,n+1}$, where $\sum_{j=1}^{n+1} p_{ij} = 1$. The general movement in all directions of particles in a 2-site G-ASIP is depicted in Figure 8:

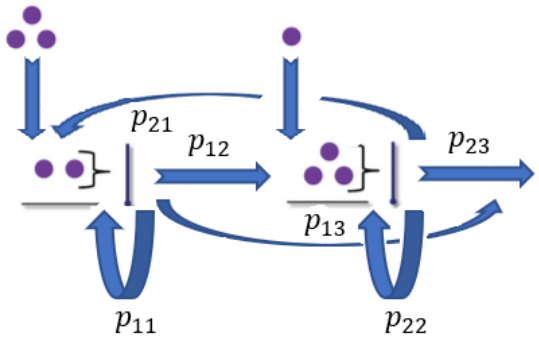


Figure 8: Particles general movement in all directions in a 2-site network: particles from site 1 can move to site 2 with probability p_{12}, or exit the system with probability p_{13}, or stay in site 1 with probability $p_{11} = 1-(p_{12}+p_{13})$. Particles from site 2 can move to site 1 with probability p_{21} or exit the system with probability m_{23} or stay in site 2 with probability $p_{22} = 1 - (p_{21} + p_{23})$. Arrivals occur at each one of the sites.

In this section we derive an explicit formula for the mean site occupancies at gate openings for the above model.

5.1 Mean sites occupancies

Calculating the mean site occupancies from the corresponding laws of motion results in

$$\begin{pmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{pmatrix} = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & & p_{n2} \\ \vdots & & \ddots & \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} E[X_1] + E[A_1] \\ E[X_2] + E[A_2] \\ \vdots \\ E[X_n] + E[A_n] \end{pmatrix}. \tag{19}$$

Define the following

$$E[\vec{X}] = \begin{pmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{pmatrix}, \quad P = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & & p_{n2} \\ \vdots & & \ddots & \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{pmatrix} \quad \textit{and} \quad E[\vec{A}] = \begin{pmatrix} E[A_1] \\ E[A_2] \\ \vdots \\ E[A_n] \end{pmatrix}. \tag{20}$$

Proposition 5.1. *The vector of mean occupancies $E[\vec{X}]$ is given by*

$$E[\vec{X}] = (I - P)^{-1} P \cdot E[\vec{A}]. \tag{21}$$

Proof. From Eq. (19) we have

$$E[\vec{X}] = P \cdot E[\vec{X} + \vec{A}]. \tag{22}$$

That is,

$$(I-P)E[\vec{X}] = P \cdot E[\vec{A}]. \tag{23}$$

□

Remark 2. *The specific movements presented in Eq. (6) and in Eq. (15) are special cases of the general movement given in Eq. (21).*

Remark 3. *Eq. (21) resembles the balance traffic equation in a general Jackson Network [7, 19] with n sites, where arrivals to site i follow a Poisson process with rate γ_i, independent of other arrival streams. Each isolated site is an $M/M/c_i$ queue. After service completion at site i, a particle moves to site j with probability q_{ij}, or leaves the network with the complementary probability $1-\sum_{j=1}^{n} q_{ij}$. The routing matrix is denoted by $Q=[q_{ij}]$ and is sub-stochastic with spectral radius less than unity. The overall arrival rate to site i is $\lambda_i = \gamma_i + \sum_{j=1}^{n} \lambda_j q_{ji}$. In matrix notation*

$$\vec{\lambda} = (I-Q^T)^{-1}\vec{\gamma}. \tag{24}$$

Note that, Eq. (21) deals with mean queue sizes, while Eq. (24) (Jackson) formula exhibits the overall rate of arrivals to each queue. Moreover, the G-ASIP model is always stable while the stability of the Jackson Network depends on the ratio between arrival and service rates.

A Special Case: Classical ASIP

In the unidirectional G-ASIP gate j opens with probability p_j, $\sum_{j=1}^{n} p_j = 1$. At a gate opening of Q_j all particles move forward to Q_{j+1}. This movement can be represented by the matrix

$$P_C = \begin{pmatrix} 1-p_1 & 0 & 0 & & 0 \\ p_1 & 1-p_2 & 0 & \cdots & 0 \\ 0 & p_2 & 1-p_3 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & p_{n-1} & 1-p_n \end{pmatrix}. \tag{25}$$

Substituting P_C in (21) yields

$$E[\vec{X}] = \begin{pmatrix} p_1 & 0 & 0 & & 0 \\ -p_1 & p_2 & 0 & \cdots & 0 \\ 0 & -p_2 & p_3 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & -p_{n-1} & p_n \end{pmatrix}^{-1} \begin{pmatrix} 1-p_1 & 0 & 0 & & 0 \\ p_1 & 1-p_2 & 0 & \cdots & 0 \\ 0 & p_2 & 1-p_3 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & p_{n-1} & 1-p_n \end{pmatrix} \cdot E[\vec{A}], \tag{26}$$

which leads to

$$E[\vec{X}] = \begin{pmatrix} \frac{1}{p_1} & 0 & 0 & & 0 \\ \frac{1}{p_2} & \frac{1}{p_2} & 0 & \cdots & 0 \\ \frac{1}{p_3} & \frac{1}{p_3} & \frac{1}{p_3} & & 0 \\ \vdots & & & \ddots & \\ \frac{1}{p_n} & \frac{1}{p_n} & \cdots & \frac{1}{p_n} & \frac{1}{p_n} \end{pmatrix} \begin{pmatrix} 1-p_1 & 0 & 0 & & 0 \\ p_1 & 1-p_2 & 0 & \cdots & 0 \\ 0 & p_2 & 1-p_3 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & p_{n-1} & 1-p_n \end{pmatrix} \cdot E[\vec{A}]. \tag{27}$$

Finally,

$$E[\vec{X}] = \begin{pmatrix} \frac{1-p_1}{p_1} & 0 & 0 & & 0 \\ \frac{1}{p_2} & \frac{1-p_2}{p_2} & 0 & \cdots & 0 \\ \frac{1}{p_3} & \frac{1}{p_3} & \frac{1-p_3}{p_3} & & 0 \\ \vdots & & & \ddots & \\ \frac{1}{p_n} & \frac{1}{p_n} & \cdots & \frac{1}{p_n} & \frac{1-p_n}{p_n} \end{pmatrix} \cdot E[\vec{A}]. \tag{28}$$

Specifically,

$$E[X_1] = \frac{1-p_1}{p_1} E[A_1], \tag{29}$$

and

$$E[X_j] = \frac{1}{p_j} \sum_{i=1}^{j-1} E[A_i] + \frac{1-p_j}{p_j} E[A_j], \quad j = 2, \ldots, n. \tag{30}$$

Eq. (28) coincides with Eq. (6.5.b) in [15].

5.1.1 Mean site occupancies for an n-site network in a homogeneous G-ASIP

In the homogeneous case where $p_j = \frac{1}{n}$, Eq. (28) becomes

$$E[\vec{X}] = \begin{pmatrix} n-1 & 0 & 0 & & 0 \\ n & n-1 & 0 & \cdots & 0 \\ n & n & n-1 & & 0 \\ \vdots & & & \ddots & \\ n & n & \cdots & n & n-1 \end{pmatrix} \cdot E[\vec{A}]. \tag{31}$$

Thus,

$$E[X_1] = (n-1)E[A_1], \tag{32}$$

and

$$E[X_j] = n \sum_{i=1}^{j} E[A_i] - E[A_j], \quad j = 2, \ldots, n. \tag{33}$$

A Special Case: Uniform particles arrival

Suppose that arrivals occur one particle at a time and the particles arrival is distributed uniformly between the sites such that $E[\vec{A}] = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$. Then, $E[\vec{X}] = (\frac{n-1}{n}, \frac{2n-1}{n}, \ldots, \frac{n^2-1}{n})^T$. Taking limit as $n \to \infty$, we obtain $\lim_{n\to\infty} E[X_j] = j$.

6 General ASIP with simultaneous openings of all gates and particles' movements in all directions

Consider an n-site G-ASIP network, where arrivals occur at all sites. Gate openings are determined by a renewal process where the time between gate openings is a random variable O. At gate opening instants all gates open simultaneously and all particles present at Q_i either move instantaneously to Q_j with probability P_{ij}, or exit the system with probability $P_{i,n+1}$, $\sum_{j=1}^{n+1} p_{ij} = 1$. We assume that the order of particles' movement is the following: First particles move outside a site and then particles move into the site. In this section we derive an explicit formula for mean site occupancies.

6.1 The $n = 2$ network

As there are 3 possible movements associated with each site, there are 3^2 possible movements in the system, leading to the following two-dimensional laws of motion.

With probabilities p_{11} in site Q_1 and p_{21} in site Q_2 (in short (p_{11}, p_{21}))

$$\begin{aligned} X_1^{(r+1)} &= X_1^{(r)} + A_1^{(r+1)} + X_2^{(r)} + A_2^{(r+1)}, \\ X_2^{(r+1)} &= 0. \end{aligned}$$

With probabilities (p_{11}, p_{22}):

$$\begin{aligned} X_1^{(r+1)} &= X_1^{(r)} + A_1^{(r+1)}, \\ X_2^{(r+1)} &= X_2^{(r)} + A_2^{(r+1)}. \end{aligned}$$

With probabilities (p_{11}, p_{23}):

$$\begin{aligned} X_1^{(r+1)} &= X_1^{(r)} + A_1^{(r+1)}, \\ X_2^{(r+1)} &= 0. \end{aligned}$$

With probabilities (p_{12}, p_{21}):

$$\begin{aligned} X_1^{(r+1)} &= X_2^{(r)} + A_2^{(r+1)}, \\ X_2^{(r+1)} &= X_1^{(r)} + A_1^{(r+1)}. \end{aligned}$$

With probabilities (p_{12}, p_{22}):

$$\begin{aligned} X_1^{(r+1)} &= 0, \\ X_2^{(r+1)} &= X_1^{(r)} + A_1^{(r+1)} + X_2^{(r)} + A_2^{(r+1)}. \end{aligned}$$

With probabilities (p_{12}, p_{23}):

$$\begin{aligned} X_1^{(r+1)} &= 0, \\ X_2^{(r+1)} &= X_1^{(r)} + A_1^{(r+1)}. \end{aligned}$$

With probabilities (p_{13}, p_{21}):

$$\begin{aligned} X_1^{(r+1)} &= X_2^{(r)} + A_2^{(r+1)}, \\ X_2^{(r+1)} &= 0. \end{aligned}$$

With probabilities (p_{13}, p_{22}):

$$\begin{aligned} X_1^{(r+1)} &= 0, \\ X_2^{(r+1)} &= X_2^{(r)} + A_2^{(r+1)}. \end{aligned}$$

With probabilities (p_{13}, p_{23}):

$$\begin{aligned} X_1^{(r+1)} &= 0, \\ X_2^{(r+1)} &= 0. \end{aligned}$$

6.1.1 Mean site occupancies

Theorem 6.1. *The mean site occupancies are identical to the mean site occupancies in the case where only one gate opens at gate-opening instants*

Proof. Taking expectations of both sides of the Laws of motion presented in Section (6.1) we get

$$\begin{pmatrix} E[X_1] \\ E[X_2] \end{pmatrix} = \begin{pmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} E[X_1] + E[A_1] \\ E[X_2] + E[A_2] \end{pmatrix}. \tag{34}$$

Denoting $P = \begin{pmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{pmatrix}$, the result coincides with Eq. (21), namely,

$$E[\vec{X}] = (I - P)^{-1} P \cdot E[\vec{A}]. \tag{35}$$

Specifically,

$$E[\vec{X}] = \frac{1}{(1-p_{11})(1-p_{22}) - p_{12}p_{21}} \begin{pmatrix} (1-p_{22})p_{11} + p_{12}p_{21} & (1-p_{22})p_{21} + p_{22}p_{21} \\ (1-p_{11})p_{22} + p_{11}p_{12} & (1-p_{11})p_{22} + p_{12}p_{21} \end{pmatrix} E[\vec{A}]. \tag{36}$$

□

6.1.2 Occupancy Load

In the elementary case $n = 1$ the total load is

$$L = E[X_1] = \frac{1}{1-p_{11}} E[A_1], \tag{37}$$

which is a geometric sum.

In the case of $n = 2$ the total load, L, is given by

$$L = E[X_1] + E[X_2] = \frac{\big(p_{11}(1-p_{22}) + p_{12}(1-p_{21})\big)E[A_1] + \big(p_{22}(1-p_{11}) + p_{21}(1-p_{12})\big)E[A_2]}{(1-p_{11})(1-p_{22}) - p_{12}p_{21}}. \tag{38}$$

Remark 4. *In the classical ASIP [1] it was shown that although the joint PGF does not admit a product form solution, the PGF of the total load does. In the current case, however, neither the site occupancy PGF nor the PGF of the total load admit a product form solution.*

The explanation is that, while in the classical ASIP movements within the first j sites don't affect the total load of those sites, in the current case, movements are in all directions, where particles can move outside the block of the first j sites and in addition particles from sites $j+1, \ldots, n$ can move backwards into the first j sites. Thus, the product form does not hold.

6.2 The n-site network

Calculating the mean site occupancies for the n-site system with simultaneous particle movements in all directions leads to

$$\begin{pmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{pmatrix} = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & & p_{n2} \\ \vdots & & \ddots & \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} E[X_1] + E[A_1] \\ E[X_2] + E[A_2] \\ \vdots \\ E[X_n] + E[A_n] \end{pmatrix}. \tag{39}$$

Eq. (39) exhibits the generality of Eq. (21), namely,

$$E[\vec{X}] = (I - P)^{-1} P \cdot E[\vec{A}], \tag{40}$$

which shows that the mean site occupancies vector in a G-ASIP network with simultaneous particle movements resemble the corresponding vector in the classical ASIP.

7 Conclusion

In this study, we extend the analysis of the n-site Asymmetric Simple Inclusion Process (ASIP) network (introduced in [1]) to a generalized ASIP (G-ASIP) framework, where particles arrive at all sites, and move in all directions at gate-opening instants, including feedback, and gate opening intervals are controlled by a renewal process. Unlike previous approaches relying on tedious successive substitution processes, we apply a more efficient matrix approach to derive the multi-dimensional probability generating functions (PGFs) of site occupancies and their means.

Three variants of particle movements are investigated: (i) unidirectional probabilistic forward movements; (ii) probabilistic forward movements combined with feedback to the first site; and (iii) general probabilistic multidirectional movements. Explicit results for mean site occupancies are derived for n-site networks, along with PGFs for small-size systems. Notably, in the case of homogeneous networks with equal probabilities of forward movements, we show that the ratio between the mean occupancies of consecutive sites converges to a constant as the number of sites grows large.

Furthermore, we explore an n-site network where all gates open simultaneously at gate opening instants, allowing particles to move in all directions. We establish that the mean site occupancies vector satisfies $E[\vec{X}] = (I - P)^{-1} P \cdot E[\vec{A}]$, where $E[\vec{X}]$ represents the mean site occupancy vector, P is the routing probability matrix corresponding to the general case, and $E[\vec{A}]$ is the vector of mean arrivals to the various sites during a random inter-opening interval O. Numerical examples are provided to illustrate the theoretical results.

This study offers further insights into the dynamics of generalized ASIP networks. It presents efficient analytical tools for analyzing complex particle movement phenomena. The findings contribute to a deeper understanding of the stochastic processes driving such network-type systems.

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8 Appendicies

A

The desired PGF $G(z_1, z_2)$ of Eq. (3) is obtained by first constructing a linear set of equations based on Eq. (2), requiring only three sequential substitutions: $z_2 = 1$, then $z_1 = 1$, and finally $z_1 = z_2$.

The above substitutions yield the set

$$\begin{cases} G(z_1,1) = & p_1m_{11}\hat{A}(z_1,1)G(z_1,1) + p_1m_{12} + p_1m_{13} + p_2m_{22}\hat{A}(z_1,1)G(z_1,1) \\ & +p_2m_{23}\hat{A}(z_1,1)G(z_1,1), \\ G(1,z_2) = & p_1m_{11}\hat{A}(1,z_2)G(1,z_2) + p_1m_{12}\hat{A}(z_2,z_2)G(z_2,z_2) + p_1m_{13}\hat{A}(1,z_2)G(1,z_2) \\ & +p_2m_{22}\hat{A}(1,z_2)G(1,z_2) + p_2m_{23}, \\ G(z_2,z_2) = & p_1m_{11}\hat{A}(z_2,z_2)G(z_2,z_2) + p_1m_{12}\hat{A}(z_2,z_2)G(z_2,z_2) + p_1m_{13}\hat{A}(1,z_2)G(1,z_2) \\ & +p_2m_{22}\hat{A}(z_2,z_2)G(z_2,z_2) + p_2m_{23}\hat{A}(z_2,1)G(z_2,1). \end{cases} \tag{41}$$

Consequently, the PGF $G(z_1,1)$ is obtained immediately from the first equation of set (41),

$$G(z_1,1) = \frac{p_1m_{12} + p_1m_{13}}{1-(p_1m_{11}+p_2)\hat{A}(z_1,1)}. \tag{42}$$

Using the two other equations in set (41), the following matrix representation is constructed

$$\begin{pmatrix} 1-(p_1m_{11}+p_1m_{13}+p_2m_{22})\hat{A}(1,z_2) & -p_1m_{12}\hat{A}(z_2,z_2) \\ -p_1m_{13}\hat{A}(1,z_2) & 1-(p_1m_{11}+p_1m_{12}+p_2m_{22})\hat{A}(z_2,z_2) \end{pmatrix} \begin{pmatrix} G(1,z_2) \\ G(z_2,z_2) \end{pmatrix} = \begin{pmatrix} p_2m_{23} \\ p_2m_{23}\hat{A}(z_2,1)G(z_2,1) \end{pmatrix}. \tag{43}$$

Define the 2×2 matrix C and the two column vectors $\vec{g}$ and $\vec{p}$ as follows

$$C = \begin{pmatrix} 1-(p_1m_{11}+p_1m_{13}+p_2m_{22})\hat{A}(1,z_2) & -p_1m_{12}\hat{A}(z_2,z_2) \\ -p_1m_{13}\hat{A}(1,z_2) & 1-(p_1m_{11}+p_1m_{12}+p_2m_{22})\hat{A}(z_2,z_2) \end{pmatrix}, \tag{44}$$

$$\vec{g} = \begin{pmatrix} G(1,z_2) \\ G(z_2,z_2) \end{pmatrix}, \quad \textit{and} \quad \vec{p} = \begin{pmatrix} p_2m_{23} \\ p_2m_{23}\hat{A}(z_2,1)G(z_2,1) \end{pmatrix}. \tag{45}$$

Then, Eq. (43) is written in a matrix notation as $C\cdot\vec{g} = \vec{p}$. Therefore, $\vec{g} = C^{-1}\cdot\vec{p}$. That is,

$$\begin{pmatrix} G(1,z_2) \\ G(z_2,z_2) \end{pmatrix} = \frac{1}{|C|} \begin{pmatrix} 1-(p_1m_{11}+p_1m_{12}+p_2m_{22})\hat{A}(z_2,z_2) & p_1m_{12}\hat{A}(z_2,z_2) \\ p_1m_{13}\hat{A}(1,z_2) & 1-(p_1m_{11}+p_1m_{13}+p_2m_{22})\hat{A}(1,z_2) \end{pmatrix} \times \begin{pmatrix} p_2m_{23} \\ p_2m_{23}\hat{A}(z_2,1)G(z_2,1) \end{pmatrix}, \tag{46}$$

leading to

$$G(1,z_2) = \frac{\big(1-(p_1m_{11}+p_1m_{12}+p_2m_{22})\hat{A}(z_2,z_2)\big)p_2m_{23} + p_1m_{12}\hat{A}(z_2,z_2)p_2m_{23}\hat{A}(z_2,1)G(z_2,1)}{\big(1-(p_1(1-m_{13})+p_2m_{22})\hat{A}(z_2,z_2)\big)\big(1-(p_1(1-m_{12})+p_2m_{22})\hat{A}(1,z_2)\big) - p_1^2m_{13}\hat{A}(1,z_2)m_{12}\hat{A}(z_2,z_2)}, \tag{47}$$

and

$$G(z_2,z_2) = \frac{p_1m_{13}p_2m_{23}\hat{A}(1,z_2) + \big(1-(p_1m_{11}+p_1m_{13}+p_2m_{22})\hat{A}(1,z_2)\big)p_2m_{23}\hat{A}(z_2,1)G(z_2,1)}{\big(1-(p_1(1-m_{13})+p_2m_{22})\hat{A}(z_2,z_2)\big)\big(1-(p_1(1-m_{12})+p_2m_{22})\hat{A}(1,z_2)\big) - p_1^2m_{13}\hat{A}(1,z_2)m_{12}\hat{A}(z_2,z_2)}. \tag{48}$$

Substituting Eq. (42) in Eq. (47) and in Eq. (48) gives

$$G(1,z_2)=\frac{\big(1-(p_1m_{11}+p_1m_{12}+p_2m_{22})\hat{A}(z_2,z_2)\big)p_2m_{23}+p_1m_{12}\hat{A}(z_2,z_2)p_2m_{23}\hat{A}(z_2,1)\frac{p_1m_{12}+p_1m_{13}}{1-(p_1m_{11}+p_2)\hat{A}(z_2,1)}}{\big(1-(p_1(1-m_{13})+p_2m_{22})\hat{A}(z_2,z_2)\big)\big(1-(p_1(1-m_{12})+p_2m_{22})\hat{A}(1,z_2)\big)-p_1^2m_{13}\hat{A}(1,z_2)m_{12}\hat{A}(z_2,z_2)}, \tag{49}$$

and

$$G(z_2,z_2)=\frac{p_1m_{13}p_2m_{23}\hat{A}(1,z_2)+\big(1-(p_1m_{11}+p_1m_{13}+p_2m_{22})\hat{A}(1,z_2)\big)p_2m_{23}\hat{A}(z_2,1)\frac{p_1m_{12}+p_1m_{13}}{1-(p_1m_{11}+p_2)\hat{A}(z_2,1)}}{\big(1-(p_1(1-m_{13})+p_2m_{22})\hat{A}(z_2,z_2)\big)\big(1-(p_1(1-m_{12})+p_2m_{22})\hat{A}(1,z_2)\big)-p_1^2m_{13}\hat{A}(1,z_2)m_{12}\hat{A}(z_2,z_2)}. \tag{50}$$

Finally, substituting Eq. (42), Eq. (49) and Eq. (50) in Eq. (2) gives the desired PGF presented in Eq. (3).

B

B.1 Mean site occupancies of an $n=2$ network (see Section 3.2)

Recalling the occupancies law of motion Eq. (2), we obtain

$$\begin{aligned}E[X_1]=&p_1m_{11}(E[X_1]+E[A_1])+p_2(E[X_1]+E[A_1])=(p_1m_{11}+p_2)E[X_1]+(p_1m_{11}+p_2)E[A_1]\\=&\frac{1-p_1(1-m_{11})}{p_1(1-m_{11})}E[A_1]=\frac{1}{p_1(1-m_{11})}E[A_1]-E[A_1],\end{aligned} \tag{51}$$

and

$$\begin{aligned}E[X_2]=&(p_1m_{11}+p_1m_{13}+p_2m_{22})(E[X_2]+E[A_2])+p_1m_{12}(E[X_1]+E[A_1]+E[X_2]+E[A_2])\\=&p_1m_{12}(E[X_1]+E[A_1])+(1-p_2m_{23})E[X_2]+(1-p_2m_{23})E[A_2].\end{aligned} \tag{52}$$

Hence,

$$E[X_2]=\frac{p_1m_{12}}{p_2(1-m_{22})}(E[X_1]+E[A_1])+\frac{1-p_2m_{23}}{p_2(1-m_{22})}E[A_2]. \tag{53}$$

Substituting Eq. (51) in Eq. (53) yields

$$E[X_2]=\frac{1}{p_2(1-m_{22})}\big(\frac{p_1m_{12}}{p_1(1-m_{11})}(E[A_1]+E[A_2])\big)-E[A_2]. \tag{54}$$

Remark 5. *A Special Case*

Substituting $m_{11}=0$, $m_{12}=1$, $m_{13}=0$, $m_{22}=0$ and $m_{23}=1$ in Eq. (51) and in Eq. (54), the results coincide with Eq. (5.5.b) in [15],

$$E[X_1]=\frac{1}{p_1}E[A_1]-E[A_1], \tag{55}$$

and

$$E[X_2]=\frac{1}{p_2}(E[A_1]+E[A_2])-E[A_2]. \tag{56}$$

B.2 Mean sites occupancies of an $n = 3$ network

Consider an $n = 3$ sites G-ASIP with probabilistic forward movements. The laws of motion are:

If gate 1 opens at step $r + 1$ then
with probability m_{11}:

$$X_1^{(r+1)} = X_1^{(r)} + A_1^{(r+1)}; \quad X_2^{(r+1)} = X_2^{(r)} + A_2^{(r+1)}; \quad X_3^{(r+1)} = X_3^{(r)} + A_3^{(r+1)};$$

with probability m_{12}:

$$X_1^{(r+1)} = 0; \quad X_2^{(r+1)} = X_1^{(r)} + A_1^{(r+1)} + X_2^{(r)} + A_2^{(r+1)}; \quad X_3^{(r+1)} = X_3^{(r)} + A_3^{(r+1)};$$

with probability m_{13}:

$$X_1^{(r+1)} = 0; \quad X_2^{(r+1)} = X_2^{(r)} + A_2^{(r+1)}; \quad X_3^{(r+1)} = X_1^{(r)} + A_1^{(r+1)} + X_3^{(r)} + A_3^{(r+1)};$$

with probability m_{14}:

$$X_1^{(r+1)} = 0; \quad X_2^{(r+1)} = X_2^{(r)} + A_2^{(r+1)}; \quad X_3^{(r+1)} = X_3^{(r)} + A_3^{(r+1)}.$$

If gate 2 opens at step $r + 1$, then
with probability m_{22}:

$$X_1^{(r+1)} = X_1^{(r)} + A_1^{(r+1)}; \quad X_2^{(r+1)} = X_2^{(r)} + A_2^{(r+1)}; \quad X_3^{(r+1)} = X_3^{(r)} + A_3^{(r+1)};$$

with probability m_{23}:

$$X_1^{(r+1)} = X_1^{(r)} + A_1^{(r+1)}; \quad X_2^{(r+1)} = 0; \quad X_3^{(r+1)} = X_2^{(r)} + A_2^{(r+1)} + X_3^{(r)} + A_3^{(r+1)};$$

with probability m_{24}:

$$X_1^{(r+1)} = X_1^{(r)} + A_1^{(r+1)}; \quad X_2^{(r+1)} = 0; \quad X_3^{(r+1)} = X_3^{(r)} + A_3^{(r+1)}.$$

If gate 3 opens at step $r + 1$ then
with probability m_{33}:

$$X_1^{(r+1)} = X_1^{(r)} + A_1^{(r+1)}; \quad X_2^{(r+1)} = X_2^{(r)} + A_2^{(r+1)}; \quad X_3^{(r+1)} = X_3^{(r)} + A_3^{(r+1)};$$

with probability m_{34}:

$$X_1^{(r+1)} = X_1^{(r)} + A_1^{(r+1)}; \quad X_2^{(r+1)} = X_2^{(r)} + A_2^{(r+1)}; \quad X_3^{(r+1)} = 0.$$

Taking expectations results in

$$\begin{pmatrix} E[X_1] \\ E[X_2] \\ E[X_3] \end{pmatrix} = \begin{pmatrix} p_1 m_{11} + p_2 + p_3 & 0 & 0 \\ p_1 m_{12} & p_1 + p_2 m_{22} + p_3 & 0 \\ p_1 m_{13} & p_2 m_{23} & p_1 + p_2 + p_3 m_{33} \end{pmatrix} \begin{pmatrix} E[X_1] + E[A_1] \\ E[X_2] + E[A_2] \\ E[X_3] + E[A_3] \end{pmatrix}, \quad (57)$$

which leads to the following matrix representation

$$\begin{aligned} \begin{pmatrix} E[X_1] \\ E[X_2] \\ E[X_3] \end{pmatrix} = & \begin{pmatrix} p_1(1 - m_{11}) & 0 & 0 \\ -p_1 m_{12} & p_2(1 - m_{22}) & 0 \\ -p_1 m_{13} & -p_2 m_{23} & p_3(1 - m_{33}) \end{pmatrix}^{-1} \\ & \times \begin{pmatrix} 1 - p_1(1 - m_{11}) & 0 & 0 \\ p_1 m_{12} & 1 - p_2(1 - m_{22}) & 0 \\ p_1 m_{13} & p_2 m_{23} & 1 - p_3(1 - m_{33}) \end{pmatrix} \begin{pmatrix} E[A_1] \\ E[A_2] \\ E[A_3] \end{pmatrix}, \end{aligned} \quad (58)$$

The last expression yields an explicit solution

$$E[X_1] = \frac{1}{p_1(1-m_{11})}E[A_1] - E[A_1], \tag{59}$$

$$E[X_2] = \frac{1}{p_2(1-m_{22})}\big(\frac{m_{12}}{1-m_{11}}E[A_1] + E[A_2]\big) - E[A_2], \tag{60}$$

$$E[X_3] = \frac{1}{p_3(1-m_{33})}\big(\frac{m_{13}(1-m_{22}) + m_{12}m_{23}}{(1-m_{11})(1-m_{22})}E[A_1] + \frac{m_{23}}{1-m_{22}}E[A_2] + E[A_3]\big) - E[A_3]. \tag{61}$$

Remark 6. *A Special Case*

Substituting $m_{11} = 0$, $m_{12} = 1$, $m_{13} = 0$, $m_{14} = 0$, $m_{22} = 0$, $m_{23} = 1$, $m_{24} = 0$, $m_{33} = 0$ *and* $m_{34} = 1$ *in Eq. (3), the results coincide with Eq. (6.5.b) in [15].*

C

We obtain the PGF $G(z_1, z_2)$ of Eq. (13) by first applying two substitutions in Eq. (11):

1. $z_2 = 1$,
2. $z_2 = z_1$.

Then, the following set of equations is obtained:

$$\begin{cases} G(z_1,1) = & p_1m_1 + \big(p_1(1-m_1) + p_2m_2\big)\hat{A}(z_1,1)G(z_1,1) + p_2(1-m_2)\hat{A}(z_1,z_1)G(z_1,z_1), \\ G(z_1,z_1) = & \big(p_1 + p_2(1-m_2)\big)\hat{A}(z_1,z_1)G(z_1,z_1) + p_2m_2\hat{A}(z_1,1)G(z_1,1). \end{cases} \tag{62}$$

The set in Eq. (62) leads to the following matrix representation

$$\begin{pmatrix} 1-(p_1(1-m_1)+p_2m_2)\hat{A}(z_1,1) & -p_2(1-m_2)\hat{A}(z_1,z_1) \\ -p_2m_2\hat{A}(z_1,1) & 1-(p_1+p_2(1-m_2))\hat{A}(z_1,z_1) \end{pmatrix} \begin{pmatrix} G(z_1,1) \\ G(z_1,z_1) \end{pmatrix} = \begin{pmatrix} p_1m_1 \\ 0 \end{pmatrix}. \tag{63}$$

Define

$$C = \begin{pmatrix} 1-(p_1(1-m_1)+p_2m_2)\hat{A}(z_1,1) & -p_2(1-m_2)\hat{A}(z_1,z_1) \\ -p_2m_2\hat{A}(z_1,1) & 1-(p_1+p_2(1-m_2))\hat{A}(z_1,z_1) \end{pmatrix}, \tag{64}$$

$$\vec{g} = \begin{pmatrix} G(z_1,1) \\ G(z_1,z_1) \end{pmatrix}, \quad \textit{and} \quad \vec{p} = \begin{pmatrix} p_1m_1 \\ 0 \end{pmatrix}. \tag{65}$$

Then, Eq. (63) is written in a matrix notation as $C \cdot \vec{g} = \vec{p}$. Therefore, $\vec{g} = C^{-1} \cdot \vec{p}$. That is,

$$\begin{pmatrix} G(z_1,1) \\ G(z_1,z_1) \end{pmatrix} = \frac{1}{|C|}\begin{pmatrix} 1-(p_1+p_2(1-m_2))\hat{A}(z_1,z_1) & p_2(1-m_2)\hat{A}(z_1,z_1) \\ p_2m_2\hat{A}(z_1,1) & 1-(p_1(1-m_1)+p_2m_2)\hat{A}(z_1,1) \end{pmatrix}\begin{pmatrix} p_1m_1 \\ 0 \end{pmatrix}, \tag{66}$$

leading to

$$G(z_1,1) = \frac{p_1m_1\big(1-(p_1+p_2(1-m_2)\hat{A}(z_1,z_1)\big)}{\big(1-(p_1+p_2(1-m_2)\hat{A}(z_1,z_1)\big)\big(1-(p_1(1-m_1)+p_2m_2)\hat{A}(z_1,1)\big) - p_2^2m_2(1-m_2)\hat{A}(z_1,z_1)\hat{A}(z_1,1)}, \tag{67}$$

and

$$G(z_1,z_1)=$$

$$\frac{p_1m_1p_2m_2\hat{A}(z_1,1)}{\big(1-(p_1+p_2(1-m_2)\hat{A}(z_1,z_1)\big)\big(1-(p_1(1-m_1)+p_2m_2)\hat{A}(z_1,1)\big)-p_2^2m_2(1-m_2)\hat{A}(z_1,z_1)\hat{A}(z_1,1)}. \tag{68}$$

Finally, substituting Eqs. (67) and Eq. (68) into Eq. (12) gives the required PGF, $G(z_1,z_2)$.

D

Proof. To prove Proposition 4.1 we substitute $p_j=\frac{1}{n}$, $m_j=m$, $j=1,\dots,n$ in Eq. (15) and obtain

$$\begin{pmatrix}E[X_1]\\E[X_2]\\\vdots\\E[X_n]\end{pmatrix}=\underbrace{\begin{pmatrix}\frac{m}{n}&\frac{m-1}{n}&\frac{m-1}{n}&\cdots&\frac{m-1}{n}\\-\frac{m}{n}&\frac{1}{n}&0&\cdots&0\\0&-\frac{m}{n}&\frac{1}{n}&&\\&\vdots&&\ddots&\\0&\cdots&0&-\frac{m}{n}&\frac{1}{n}\end{pmatrix}^{-1}}_{(I-P)^{-1}}\underbrace{\begin{pmatrix}1-\frac{m}{n}&\frac{1-m}{n}&\frac{1-m}{n}&\cdots&\frac{1-m}{n}\\\frac{m}{n}&1-\frac{1}{n}&0&\cdots&0\\0&\frac{m}{n}&1-\frac{1}{n}&&\\&\vdots&&\ddots&\\0&\cdots&0&\frac{m}{n}&1-\frac{1}{n}\end{pmatrix}}_{P}\begin{pmatrix}E[A_1]\\E[A_2]\\\vdots\\E[A_n]\end{pmatrix}. \tag{69}$$

In matrix notation,

$$E[\vec{X}]=(I-P)^{-1}P\cdot E[\vec{A}]. \tag{70}$$

Define,

$$C=\begin{pmatrix}\frac{n-m^n}{m^n}&\frac{n(1-m^{n-1})}{m^n}&\frac{n(1-m^{n-2})}{m^n}&\cdots&\frac{n(1-m)}{m^n}\\\frac{n}{m^{n-1}}&\frac{n-m^{n-1}}{m^{n-1}}&\frac{n(1-m^{n-2})}{m^{n-1}}&\cdots&\frac{n(1-m)}{m^{n-1}}\\\frac{n}{m^{n-2}}&\cdots&\frac{n-m^{n-2}}{m^{n-2}}&\cdots&\frac{n(1-m)}{m^{n-2}}\\&\vdots&&\ddots&\\\frac{n}{m}&\cdots&&\frac{n}{m}&\frac{n-m}{m}\end{pmatrix}. \tag{71}$$

In order to complete the proof, it is required to show that $(I-P)^{-1}\cdot P=C$. Instead of calculating $(I-P)^{-1}$, we show that $P=(I-P)\cdot C$. Indeed,

$$\begin{aligned}&\begin{pmatrix}1-\frac{m}{n}&\frac{1-m}{n}&\frac{1-m}{n}&\cdots&\frac{1-m}{n}\\\frac{m}{n}&1-\frac{1}{n}&0&\cdots&0\\0&\frac{m}{n}&1-\frac{1}{n}&&\\&\vdots&&\ddots&\\0&\cdots&0&\frac{m}{n}&1-\frac{1}{n}\end{pmatrix}\\&=\begin{pmatrix}\frac{n-m^n}{m^n}&\frac{n(1-m^{n-1})}{m^n}&\frac{n(1-m^{n-2})}{m^n}&\cdots&\frac{n(1-m)}{m^n}\\\frac{n}{m^{n-1}}&\frac{n-m^{n-1}}{m^{n-1}}&\frac{n(1-m^{n-2})}{m^{n-1}}&\cdots&\frac{n(1-m)}{m^{n-1}}\\\frac{n}{m^{n-2}}&\cdots&\frac{n-m^{n-2}}{m^{n-2}}&\cdots&\frac{n(1-m)}{m^{n-2}}\\&\vdots&&\ddots&\\\frac{n}{m}&\cdots&&\frac{n}{m}&\frac{n-m}{m}\end{pmatrix}\begin{pmatrix}\frac{m}{n}&\frac{m-1}{n}&\frac{m-1}{n}&\cdots&\frac{m-1}{n}\\-\frac{m}{n}&\frac{1}{n}&0&\cdots&0\\0&-\frac{m}{n}&\frac{1}{n}&&\\&\vdots&&\ddots&\\0&\cdots&0&-\frac{m}{n}&\frac{1}{n}\end{pmatrix}.\end{aligned} \tag{72}$$

This completes the proof. □

E

Proof. To prove Corollary 4.1.1 we explicitly solve Eq. (16) and obtain

$$E[X_1] = \sum_{k=1}^{n} \frac{n(1-m^{n-k+1})}{m^n} E[A_k], \tag{73}$$

and

$$E[X_j] = \sum_{k=1}^{j-1} \frac{n}{m^{n-j+1}} E[A_k] + \sum_{k=j}^{n} \frac{n(1-m^{n-k+1})}{m^{n-j+1}} E[A_k], \quad j = 2, \ldots, n. \tag{74}$$

Thus, the ratio for $j = 1, \ldots, n$ is

$$\frac{E[X_{j+1}]}{E[X_j]} = \frac{\sum_{k=1}^{j} \frac{n}{m^{n-j}} E[A_k] + \sum_{k=j+1}^{n} \frac{n(1-m^{n-k+1})}{m^{n-j}} E[A_k]}{\sum_{k=1}^{j-1} \frac{n}{m^{n-j+1}} E[A_k] + \sum_{k=j}^{n} \frac{n(1-m^{n-k+1})}{m^{n-j+1}} E[A_k]}. \tag{75}$$

Therefore,

$$\frac{E[X_{j+1}]}{E[X_j]} = m \frac{\sum_{k=1}^{j} E[A_k] + \sum_{k=j+1}^{n} (1-m^{n-k+1}) E[A_k]}{\sum_{k=1}^{j-1} E[A_k] + \sum_{k=j}^{n} (1-m^{n-k+1}) E[A_k]}. \tag{76}$$

When $n \to \infty$

$$\lim_{n\to\infty} \frac{E[X_{j+1}]}{E[X_j]} = m \frac{\sum_{k=1}^{\infty} E[A_k]}{\sum_{k=1}^{\infty} E[A_k]} = m. \tag{77}$$

and $E[X_j] = m^{j-1} E[X_1]$.

□

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